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Statistical mechanics of topologically constrained systems: application to self-organizing diffusion in plasmas

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In a standard diffusion process, the distribution of particles progressively approaches a flat density profile. Such flattening results in the maximization of the entropy of the system. The scenario is different if the diffusing particles are subjected to a set of constraints: while the entropy increases, density may self-organize into a heterogeneous distribution reflecting the constrained nature of the statistical process.

A well-known example is the *inward* (or *up-hill*) *diffusion* of charged particles in dipole magnetic fields, such as the magnetosphere of the Earth. Here, the first adiabatic invariant is preserved throughout the diffusion process caused by random fluctuations in the electric field. Thus, the first adiabatic invariant acts as a *topological constraint*, whose spatial inhomogeneity is responsible for the resulting peaked density profile [1-2].

In this study, we consider the problem of statistical mechanics in topologically constrained systems. We identify the possible mechanisms of self-organization, and show that the form of the equilibrium distribution function depends on the integrability of the constraints. Integrable constraints preserve the Hamiltonian structure of dynamics, and appear as Casimir invariants spanning the center of the Poisson algebra. Then, the Boltzmann distribution of equal probability on constant energy surfaces is achieved on each Casimir leaf, each of them representing a symplectic submanifold [3-5].

On the other hand, non-integrable constraints break the Hamiltonian structure of dynamics; well-known examples are non-holonomic constraints on Lagrangian mechanics. Another important example pertaining to plasma physics is the motion of charged particles in finite-current magnetic fields. Under such conditions, the standard formulation of statistical mechanics (which relies on the invariant phase space volume provided by Liouville's theorem of Hamiltonian mechanics) does not apply. Recalling that Hamiltonian systems are endowed with field tensors (antisymmetric operators) that satisfy the Jacobi identity (and hence have zero 'helicity'), our aim is to extend the scope toward the class of systems governed by finite-helicity field tensors [6].

Here we introduce a new class of field tensors, *Beltrami operators*, that are characterized by Beltrami vector fields: in 3 dimensions, a Beltrami operator is represented by a vector field aligned with its own curl. We prove an H theorem for this Beltrami class.

The most general class of energy-conserving systems are non-Beltrami, for which we identify the "*field charge*" that is responsible of a novel type of self-organization uniquely associated to non-integrable constraints. This type of heterogeneity does not rely on the existence of integral invariants, but on an intrinsic distortion of space that cannot, in general, be removed by a suitable coordinate change. The essence of the theory can be delineated by classifying three-dimensional dynamics. We then generalize to systems of arbitrary (finite) dimensions.

As an application to plasma physics, we consider diffusion by ExB drift in an ensemble of charged particles permeated by a non-vacuum magnetic field, and perform numerical simulations. Figure 1(a) shows the profile of the field charge characterizing a non-vacuum magnetic field. Figure 1(b) shows the corresponding equilibrium density distribution.

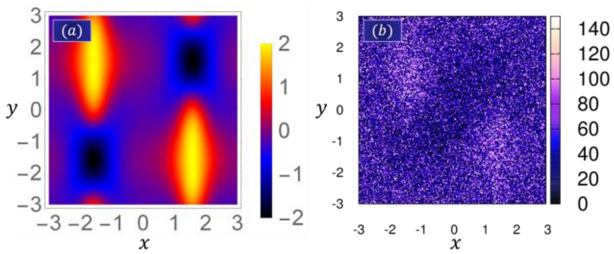

Figure 1. (a) Density plot of field charge. (b) Calculated equilibrium density distribution. Figures from [6].

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