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Magnetic moment of thermal plasma: Revisiting the Bohr-van Leeuwen theory

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Magnetic properties of thermal plasma can be explained by studying behavior of charged particles constituting the plasma in the magnetic field. According to the Bohr-van Leeuwen theory, it has been believed that the thermal plasma do not have magnetic moment. Here, we revisit the foundation of the theory and show that the thermal plasma do have magnetic moment and it plays important roles in the solar atmosphere. It is suggested that the magnetic Kelvin force acting on the magnetized plasma should be include in the magneto hydrodynamic (MHD) equation of motion.

We follow the similar procedure as Bohr and van Leeuwen used [1]. We start with the Hamiltonian of an ensemble of charged particles numbered by "i". The Cartesian coordinates $\mathbf{r}_i = (x_i, y_i, z_i)$ are used. We assume that the magnetic field is uniform (or slowly varying in space and time) and directed along z, and no scalar potential exists. The corresponding vector potential is $\mathbf{A} = (1/2)\mathbf{B}\times\mathbf{r}$, and the canonical momenta are,

$$P_{xi} = m_i\dot{x}_i - \frac{B}{2}q_i y_i,\ P_{yi} = m_i\dot{y}_i + \frac{B}{2}q_i x_i,\ P_{zi} = m_i\dot{z}_i.$$

The Hamiltonian is,

$$\begin{aligned}\mathcal{H} &= \sum_i \frac{1}{2}m_i\left(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2\right)\\ &= \sum_i \frac{1}{2}m_i\left\{\left(\frac{P_{xi}}{m_i} + \frac{Bq_i}{2m_i}y_i\right)^2 + \left(\frac{P_{yi}}{m_i} - \frac{Bq_i}{2m_i}x_i\right)^2 + \left(\frac{P_{zi}}{m_i}\right)^2\right\}.\end{aligned}$$

As is shown in the first line, the Hamiltonian does not depend on the magnetic field strength numerically. This is because the Lorentz force does not modify energy. However, when we use the canonical momenta as is shown in the second line, the Hamiltonian depends on the magnetic field. By using this expression, we can derive the magnetic moment,

$$\boldsymbol{\mu} = -\frac{\partial\mathcal{H}}{\partial B}\mathbf{b} = \sum_i \frac{q_i}{2}\left(x_i\dot{y}_i - \dot{x}_i y_i\right)\mathbf{b} = \sum_i \frac{q_i}{2}\mathbf{r}_{\perp i}\times\mathbf{v}_{\perp i},$$

where $\mathbf{b}$, $\mathbf{r}_{\perp i}$, and $\mathbf{v}_{\perp i}$, are the unit vector along the magnetic field, the position and the velocity components perpendicular to the magnetic field respectively. If we calculate the ensemble average of this magnetic moment using the canonical distribution function, we will find null values as is shown by van Vleck [1]. This is because the average velocity at any point in the coordinate space is zero due to random thermal motion of particles. The magnetic moment is the result of the curved motion of charged particles in the magnetic field, but this fact is not reflected in the distribution function. The curved motion is one of the accelerated motion, even though the energy does not change. To understand the accelerated motion, it is necessary to solve the equation of motion. We use the Hamiltonian equation of motion.

$$\dot{P}_{xi} = -\frac{\partial\mathcal{H}}{\partial x_i},\ \dot{P}_{yi} = -\frac{\partial\mathcal{H}}{\partial y_i},\ \dot{P}_{zi} = -\frac{\partial\mathcal{H}}{\partial z_i}$$

For the particles satisfying the equation of motion, the position of each particle is determined by its velocity. This relation is used to calculate the magnetic moment.

$$\mathbf{r}_{\perp i} = -\frac{m_i}{q_i B}\mathbf{v}_{\perp i}\times\mathbf{b},\ \ \boldsymbol{\mu} = -\sum_i \frac{m_i \mathrm{v}_{\perp i}^2}{2B}\mathbf{b}$$

This equation tells us that the magnetic moment of the plasma is the sum of thermal energies of particles in the perpendicular plane, divided by the magnetic field strength, and directed anti-parallel to the field. Here we assume that the plasma consists of equal number density N of electrons and protons and they are in thermal equilibrium with the temperature T. The averaged magnetic moment per unit volume is,

$$\mathbf{M} = -\frac{2Nk_BT}{B}\mathbf{b} = -\frac{P}{B}\mathbf{b},\ \ P = 2Nk_BT.$$

The thermal plasma is a highly nonlinear diamagnetic media. It is necessary to introduce this expression into the equation $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$ for studies of magnetic properties of the plasma.

The force acting on the magnetic moment in non-uniform magnetic field is called the magnetic Kelvin force. In the present case, the Kelvin force is,

$$\mathbf{F}_K = (\mathbf{M}\cdot\nabla)\mathbf{B} = -\frac{P}{B}(\nabla_{\parallel}B)\mathbf{b}.$$

This force corresponds to the mirror force in case of the adiabatic treatment. This force should be added to the MHD equation of motion. The revised MHD equation of motion along the field is as follows.

$$\rho_m\left\{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial s}\right\} = -\frac{\partial P}{\partial s} - \frac{P}{B}\frac{\partial B}{\partial s} - \rho_m \mathrm{g}\cdot\cos(\theta) - \alpha\rho_m u.$$

Generally, magnetic field strength weakens upwards in the solar atmosphere, hence F_K is directed upward. When F_K exceeds the gravity force, plasma is pushed upward and is resulted in loop-top concentration of plasma in closed magnetic loops. In case of open magnetic field, temperature dependent plasma up-flows are expected. Both of these phenomena are often observed in the solar atmosphere, but not yet explained satisfactory.

Ott et al. [2] showed numerical simulations of crystallization that appears to be in conflict with the Bohr-van Leeuwen theory.

References

[1] J. H. van Vleck, *The Theory of Electric and Magnetic Susceptibilities*, (Oxford Press, 1932).
[2] T. Ott, H. Lowen, and M. Bonitz, Phys. Rev. Lett. **111**, 065001 (2013)