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Discrete enstrophy levels and the relaxation process: Self-organizationof zonal flow in the view of variational principle

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1. Outline

The partition of enstrophy between zonal (ordered) and wavy (turbulent) components of vorticity has been studied for the beta-plane model of two-dimensional barotropic flow. An analytic estimate of the minimum value for the zonal component has been derived. The energy, angular momentum, circulation, as well as the total enstrophy are invoked as constraints for the minimization of the zonal enstrophy. The corresponding variational principle has an interesting mathematical structure in that the target functional, the zonal enstrophy is not coercive with respect to the norm of enstrophy, by which the constraints work differently than usual variational principles. A discrete set of zonal enstrophy levels is generated by the energy constraint; each level is specified by an eigenvalue that represents the lamination period of zonal flow. However, the value itself of the zonal enstrophy level is a function of only angular momentum and circulation, being independent of the energy (and total enstrophy). Instead, the energy works in selecting the "level" (eigenvalue) of the relaxed state. The relaxation occurs by emitting small-scale wavy enstrophy, and continues as far as the nonlinear effect, scaled by the energy, can create wavy enstrophy. Comparison with numerical simulations shows that the theory gives a proper estimate of the zonal enstrophy in the relaxed state.

2. Formulation

2.1. Vorticity on a beta-plane

We consider barotropic flow on a beta-plane:

$$\Omega = \{\boldsymbol{z} = (x,y)^T\,;x \in [0,1), y \in [0,1)\}.$$

Here, x is the longitude and y is the latitude. The state variable is the fluid vorticity ω. We define the Gauss potential φ by $-\Delta\varphi = \omega$ or $\varphi = K\omega$ $(K = (-\Delta)^{-1})$. On x=0 and 1, we impose periodix boundary condition, while on y=0 and 1, Dirichlet boundary condition $\varphi|_{y=0} = \varphi|_{y=1} = 0$.

Taking into account the Coriolis force, the governing equation of ω is

$$\partial_t\omega + \{\omega + \beta y, \varphi\} = 0, \qquad (1)$$

where $\{f,g\} = (\partial_x f)(\partial_y g) - (\partial_x g)(\partial_y f)$, and β is a real constant number measuring the meridional variation of the Coriolis force. We use the following functionals which are constants of motion of (1).

Total enstrophy: $Q(\omega) = \frac{1}{2}\|\omega\|^2$

Energy: $E(\omega) = \frac{1}{2}\langle\varphi,\omega\rangle$

Circulation: $F(\omega) = \langle 1,\omega\rangle$

Angular-momentum: $L(\omega) = \langle y,\omega\rangle$

We denote $\langle f,g\rangle = \int_\Omega f(z)g(z)\,d^2z$ and $\|f\| = \langle f,f\rangle$.

2.2. Zonal and wavy components

We say ω is zonal when $\partial_x\omega \equiv 0$ in Ω and define zonal component of ω as $\omega_z := \int \omega\,dx$. Also we define wavy component as $\omega_w := \omega - \omega_z$.

3. Zonal enstrophy

Here, we take into account the constraint by total enstrophy, circulation, angular-momentum and energy, and seek the minimizer of

$$Z(\omega) - \nu Q(\omega) - \mu_0 F(\omega) - \mu_1 L(\omega) - \mu_2 E(\omega). \qquad (2)$$

Here, we denote $Z(\omega) = Q(\omega_z)$, which is zonal enstrophy, and ν, μ_0, μ_1, μ_2 is a Lagrange multiplier.

Then, we solve the Eular-Lagrange equation and obtain

$$Z_\lambda = \frac{A_1^2\lambda^3}{8}(2\lambda + \sin 2\lambda) + \frac{A_2^2\lambda^3}{8}(2\lambda - \sin 2\lambda) + \frac{A_1A_2\lambda^3}{4}(1 - \cos 2\lambda). \qquad (3)$$

Here, A_1, A_2 is determined by the constraints and λ is eigenvalues that represent the lamination period of zonal flow. The local minimums of Z_λ represents the local minimums of Zonal enstrophy and corresponding λ are the eigenvalues. Fig.1(left) shows the graph of Z_λ.

4. Conclusion

We have found a discrete set of zonal enstrophy levels that are quantized by the eigenvalue λ measuring the lamination period. The energy constraint plays an essential role in creating the discrete zonal enstrophy levels. Interestingly, the value of the energy does not influence the value of each zonal enstrophy, which is determined only by the other constants circulation and angular momentum. Instead, the value of energy works in selecting the eigenvalue λ of the relaxed state. This unusual phenomenon in variational principle is cause by the con-coerciveness of the target functional $Z(\omega)$ with respect to the norm $\|\omega\|$.

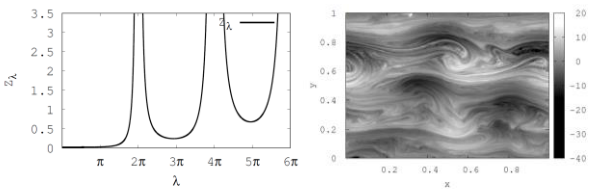

Figure1 (left) The graph of Z_λ. Each local minimums represents the local minimums of Zonal enstrophy. (right) Zonal flow self-organized in simulation (gray level corresponds the local value of ω).

Reference

H. Aibara and Z. Yoshida. "Partition of enstrophy between zonal and turbulent components." arXiv preprint arXiv:2002.08592 (2020).