6th Asia-Pacific Conference on Plasma Physics, 9-14 Oct, 2022, Remote e-conference

Numerical methods for calculating statistical equilibria of two-dimensional turbulence considering all Casimir invariants.

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1. Introduction

The Euler equation of two-dimensional fluid has a similar form to the Vlasov equation that describes the motion of thin plasma laid in the uniform magnetic field perpendicular to it. Also, the Euler equation can be considered as a limit of infinite Rossby deformation radius of the Charney-Hasegawa-Mima equation. It is known that coherent structures arise in two-dimensional turbulence. The Miller-Robert-Sommeria theory (MRS theory) is one of the theories that attempt to explain the appearance of such coherent structures using statistical mechanics[1,2]. In this theory, the statistical equilibrium, which is expected to correspond to the coherent structure, is obtained by maximizing the entropy of vorticity mixing. However, the optimization problem of the entropy is difficult to solve numerically, since the conservations of all Casimir invariants and the energy need to be considered. Especially, when the symmetry that the initial vorticity field has is broken, the calculation of statistical equilibria is impossible in many cases with the methods proposed in previous studies. We seek to resolve this difficulty by developing new methods[3].

2. Numerical method

We consider two-dimensional flow on the unit sphere. We assume the initial vorticity field that consists of K levels of vorticity patches, and let Q_k and S_k $(k = 1, \cdots, K)$ be the vorticity value and the area of the k-th vorticity patch, respectively. For a point on the sphere, let λ and μ denote the longitude and the sine of the latitude of the point, respectively, and we use (λ, μ) as the coordinate system on the sphere. We define r_{ijk} to be the proportion of the area occupied by the k-th patch in the infinitesimal neighborhood of the computational grid $(\lambda_i, \mu_j)(1 \le i \le I, 1 \le j \le J)$. By introducing this proportion, we define the macroscopic vorticity field $\overline{q}_{ij} = \sum_{k=1}^{K} Q_k r_{ijk}$. The mixing entropy is given by

$$S_{\mathrm{mix}} = -\frac{1}{2}\sum_{i,j,k} w_j r_{ijk} \log r_{ijk},$$

where w_j is the Gaussian weight of the latitudinal grid μ_j. The statistical equilibrium is defined as the state in which S_{mix} attains the maximum value under the linear and nonlinear constraints. The linear constraints come from the incompressibility of the fluid, the nature that the sum of the proportions is unity, and the conservation of the angular momentum. The nonlinear constraint is the energy conservation law. The search of the statistical equilibrium, however, tends to be trapped to a narrow neighborhood of a saddle of S_{mix}. To conduct more global search, we transform the problem so that the problem becomes geometrically tractable. We do not take r_{ijk}'s as variables, but the multi-dimensional point of spectral coefficients $Z = (\zeta_{m,n})$ $(|m| \le n \le N)$ of the macroscopic vorticity field as the variable. Note that we now consider the expansion of the macroscopic vorticity field by the spherical harmonics $Y_{m,n}$ with the truncation wavenumber N. Here, r_{ijk}'s are determined so as to maximize the mixing entropy S_{mix} under the linear constraints of the original problem and the following constraint

$$\sum_{k=1}^{K} Q_k r_{ijk} = \overline{q}_{ij} = \sum_{|m|\le n\le N} \zeta_{m,n} Y_{m,n}(\lambda_i, \mu_j)$$

for a given Z. The point Z must be inside a bounded domain P in the space of Z because of the inequality $r_{ijk} > 0$. The intersection of P and the level surface of energy E is the set in which we have to search for the statistical equilibrium. By generating a number of initial points to start searching, an extensive search can be conducted. We define a convex function $B(Z)$ which has finite values in the interior of P but goes to infinity as approaching the boundary of P. The level surface $B(Z) = c$ for a large $c > 0$ mimics the boundary of P. We choose a linear function arbitrarily and find its maximum point on this surface. Then we pull the point down to the energy level surface of the initial energy value to generate initial points to start searching. The entropy optimization is done by using the projection gradient method on the energy level surface.

3. Results

We calculated the statistical equilibrium for the following two zonal initial vorticity fields,
(a): $q(\mu) = 12 \times 0.07 P_3(\mu) + 2\mu$,
and (b): $q(\mu) = -12 \times 0.07 P_3(\mu) + 2\mu$, respectively, by approximating the profiles by 32 levels of vorticity patches. Here, $P_3(\mu)$ is the Legendre polynomial of degree 3, the L^2 norm of which is normalized to $\sqrt{2}$. The statistical equilibria of wavenumber 2 type, which has a similar vorticity field structure to that of the final state of time evolution of the vorticity equation from the initial field in both cases (a) and (b). The equilibria of wavenumber 1 type were also obtained. In the case (a), the wavenumber 2 equilibrium was a saddle of the entropy whereas the wavenumber 2 equilibrium was the (local) maximum in the case (b).

References

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