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Relativistic Effects on Plasma Enstrophy: Unveiling the Role of Clebsch Variables

K. Takeda[1,2] and N. Sato[2]
[1] Graduate School of Frontier Sciences, The University of Tokyo,
[2] National Institute for Fusion Science
e-mail (speaker): nunotani.keiichiro20@ae.k.u-tokyo.ac.jp

High-energy astrophysics increasingly relies on understanding relativistic plasmas. Enstrophy, a key quantity characterizing the structure of plasma flow, can be generalized for relativistic scenarios [1]. This relativistic enstrophy considers the relativistic time dilation in the integration domain, called the s-plane. It is a curved hypersurface in space-time reflecting the intrinsic time s of each fluid element (see figure 1b). An analogous construction applies to relativistic helicity [2].

Introducing potentials φ, λ^1, σ_1, λ^2, and σ_2, the velocity field V is represented as

$$mV = \nabla\varphi + \lambda^1\nabla\sigma_1 + \lambda^2\nabla\sigma_2. \qquad (1)$$

This expression is called a Clebsch representation, and the potential fields above are called Clebsch potentials or Clebsch variables [3]. Mathematically, a general three-dimensional vector field can be cast into the form of equation (1) [4].

The problem with the relativistic enstrophy defined in [1] is that it is difficult to evaluate explicitly. This issue is severe for complex flows, which are more likely to violate conservation of conventional (non-relativistic) enstrophy. Indeed, in order to find the shape of the s-plane, it is necessary to know how much proper time has elapsed for each fluid element, which means that all orbits (the trajectories that individual fluid particles follow) need to be obtained. The orbits can be obtained analytically only for highly symmetric flows, which allow for a significant simplification of the governing equations. On the other hand, for highly symmetric flows, it is known that relativistic effects do not appear in conventional enstrophy. Therefore, a practical formulation of relativistic enstrophy that can be evaluated even for complicated flows is desirable.

In this work, we formulate both relativistic helicity and relativistic enstrophy by pursuing a different approach, in which the domain of integration is not an s-plane but a t-plane (see figure 1a). The relativistic helicity is defined for a three-dimensional fluid as

$$\int_\Omega \boldsymbol{u}\cdot\boldsymbol{\omega}\; d^3x, \qquad (2)$$

where $\boldsymbol{u} = \gamma\boldsymbol{v}$ denotes the spatial components of the four-velocity, $\boldsymbol{v} = d\boldsymbol{x}/dt$, γ is the Lorentz factor, and the relativistic vorticity is given by $\boldsymbol{\omega} = \nabla\times\boldsymbol{u}$. Here, $\Omega\subset\mathbb{R}^3$ is a time-independent domain. The relativistic enstrophy is defined for a two-dimensional fluid as

$$\int_\Sigma \frac{\omega_z^2}{2\gamma\varrho}\; d^2x \qquad (3)$$

with ϱ the relativistic mass density and $\omega_z = \boldsymbol{\omega}\cdot\nabla z$. Here, integration is carried out over a time-invariant domain $\Sigma\subset\mathbb{R}^2$.

In our study, we also find that the relativistic helicity (2) and the relativistic enstrophy (3) occur as Casimir invariants of a relativistic noncanonical Hamiltonian theory. The Clebsch representation of the relativistic flow plays a critical role in the identification of such Hamiltonian structure.

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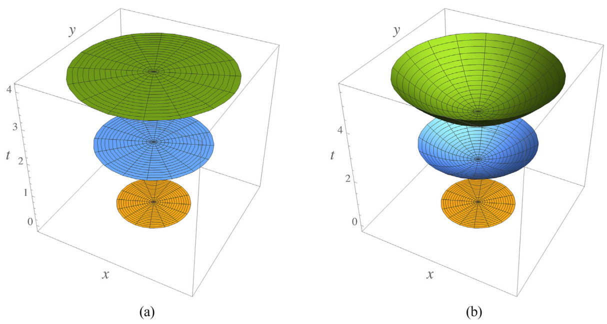

Figure 1. A conceptual drawing of t-plane (left) and s-plane (right) of the fluid flowing outward radially from the origin. The t-plane is a hyperplane in space-time at constant time t. On the other hand, the s-plane is a curved hypersurface in space-time reflecting the intrinsic time s of the fluid. In this figure, the fluid farther from the origin is faster, so more time t has passed on the outer side [1].