Convergence Rate of Multi-region Relaxed MHD Equilibria to Ideal MHD Equilibria

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A common strategy for designing magnetic confinement devices involves solving the ideal magnetohydrodynamics (MHD) equilibrium equations. Consequently, understanding the limitations of computational approximations to ideal MHD equilibria is of practical importance. Several effective computational tools have been developed to approximate these equilibria.

One such method discretizes a toroidal domain into multiple regions and seeks Taylor-relaxed states within each. This leads to the formulation of the multi-region relaxed magnetohydrodynamics (MRxMHD) equilibrium equations [1]. This approach underpins computational tools such as the Stepped Pressure Equilibrium Code (SPEC) [2]. Previous work has shown that, in the limit where the number of regions becomes infinite, an MRxMHD equilibrium satisfies the ideal MHD equilibrium equations, established through the corresponding variational principles [3].

In this work, we quantify the rate at which the objective function for MRxMHD equilibria converges to that of ideal MHD equilibria as the number of regions increase. Specifically we consider Λ, a solid torus embedded in $\mathbb{R}^3$ and a smooth function $\varphi:\Lambda\rightarrow[0,1]$ whose level sets $\varphi^{-1}(c)$ are toroidal surfaces for each $c\in(0,1]$ and $\varphi^{-1}(0)$ is a line. Define $V(c)$ as the volume enclosed by the level set $\varphi^{-1}(c)$ excluding $\varphi^{-1}(0)$, for $c\in(0,1]$. A selected poloidal and toroidal cross section of $V(c)$ are denoted $\Upsilon_P(c)$ and $\Upsilon_T(c)$, respectively.

Consider a smooth, divergence-free vector field B on Λ, tangential to the level sets of φ, with both B and its first derivatives square-integrable. The ideal MHD equilibrium equations are necessary and sufficient conditions to find a (B, φ) that are critical points of the energy functional

$$L_{\mathrm{MHD}}(B,\varphi) = \int_\Lambda \left(\frac{1}{2}B\cdot B - p(\varphi)\right)dV \quad (1)$$

given that the flux of B through $\Upsilon_P(c)$ and $\Upsilon_T(c)$ are prescribed smooth functions of $c\in(0,1]$. Additionally, the pressure p is given as a smooth function of $c\in[0,1]$. In equation (1) dV is a volume element.

We now focus on the MRxMHD framework, in which the domain Λ is partitioned into a finite number of nested toroidal subregions Λ_i, i = 1 to n (see Figure 1). Each subregion Λ_i contains a poloidal cross-section $\Upsilon_{i,P}$, and for i>1, includes a toroidal cross-section $\Upsilon_{i,T}$.

Consider a smooth, divergence-free vector field B_i on Λ_i, tangential to the boundary $\partial\Lambda_i$, with both B_i and its first derivatives square-integrable for each i. The MRxMHD equilibrium equations are necessary and sufficient to determine critical points of

$$L_{\mathrm{MRxMHD}}(B_1,\ldots,B_n,\Lambda_1,\ldots,\Lambda_n) = \sum_{i=1}^{n}\int_{\Lambda_i}\left(\frac{1}{2}B_i\cdot B_i - p_i\right)dV \quad (2)$$

where the flux of B_i through $\Upsilon_{i,P}$ and $\Upsilon_{i,T}$, each pressure p_i, and the helicity of each B_i are prescribed constants. We select values for these constants by discretizing the smooth pressure and flux profiles from the MHD case, together with the ideal MHD equilibrium equations.

We show that as the number of regions increases, under appropriate assumptions on existence and differentiability, the convergence rate at critical points is bounded by

$$|L_{\mathrm{MHD}}(B,\varphi) - L_{\mathrm{MRxMHD}}(B_1,\ldots,B_n,\Lambda_1,\ldots,\Lambda_n)| \leq \mathcal{O}(1/n) \quad (3)$$

when, for each i, the boundaries $\varphi^{-1}(i/n)\subseteq\partial\Lambda_i$, and $B_i=B$ on $\varphi^{-1}(i/n)$. This illustrates how numerical schemes such as SPEC approximately solve the variational problem for ideal MHD equilibria, given suitable choices of flux, helicity, and pressure.

References
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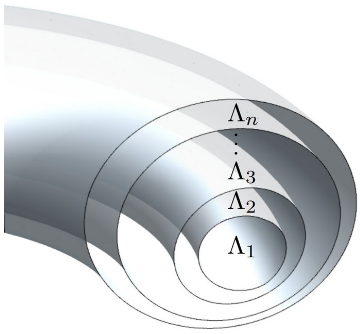

Figure 1: A cross section of a toroidal region partitioned by toroidal surfaces.