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Helically symmetric equilibria of incompressible MHD in cylindrical geometry

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A Hamiltonian structure of single helicity and incompressible magnetohydrodynamics (MHD) in a cylindrical geometry was clarified in [1]. Here, the single helicity dynamics means that physical quantities have the following spatio-temporal dependence

$$f(r,\theta,z,t) = \sum_{\ell=-\infty}^{\infty} f_\ell(r,t)\mathrm{e}^{\mathrm{i}\,\ell(M\theta+N\zeta)}, \qquad (1)$$

where f is an arbitrary physical quantity, (r,θ,z) are the cylindrical coordinates, $\zeta := z/R_0$ with $2\pi R_0$ being the length of the plasma column, M and N are the principal poloidal and toroidal mode numbers, respectively, and ℓ represents their harmonics. A coordinate $\alpha := M\theta/K + z$ with $K := N/R_0 \neq 0$ expresses the phase as $\ell(M\theta+N\zeta) = \ell K\alpha$. Then f depends on r, α and t only.

The incompressible fluid velocity $\boldsymbol{u}$ and magnetic field $\boldsymbol{B}$ are expressed as

$$\boldsymbol{u} = \boldsymbol{h}\times\nabla\varphi + u_h\boldsymbol{h}, \qquad (2)$$

$$\boldsymbol{B} = \nabla\psi\times\boldsymbol{h} + B_h\boldsymbol{h}, \qquad (3)$$

where φ, u_h, ψ and B_h are functions of r, α and t, and

$$\boldsymbol{h} := \frac{1}{K_0^2r^2}(-Kr\hat{\boldsymbol{\theta}} + M\hat{\boldsymbol{z}}) \qquad (4)$$

is an incompressible vector field. Here, $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{z}}$ are unit vectors in the θ and z directions, respectively, and $K_0^2r^2 := M^2 + K^2r^2$.

Appropriate phase-space variables for this system were found to be $v = (v^1,v^2,v^3,v^4) := (U,u_h,\psi,B_h^\star)$ with $U := \nabla\cdot(|\boldsymbol{h}|^2\nabla\varphi) =: \mathcal{L}\varphi$ and $B_h^\star := gB_h - f\psi$, where $f(r) := \boldsymbol{h}\cdot\nabla\times\boldsymbol{h} = -2MK/(K_0^2r^2)^2$ and $g(r) := |\boldsymbol{h}|^2 = 1/(K_0^2r^2)$. The Hamiltonian and the Poisson tensor is given by

$$H[v] := \frac{1}{2}\int \mathrm{d}V\,\Big(-U(\mathcal{L}^{-1}U) + gu_h^2 - \psi(\mathcal{L}\psi) + \frac{1}{g}\left(B_h^\star + f\psi\right)^2\Big), \qquad (5)$$

$$\mathcal{J} := \begin{pmatrix} [\circ, U+fu_h] & [\circ,u_h] & [\circ,\psi] & [\circ, gB_h] \\ [\circ,u_h] & 0 & 0 & [\circ,\psi] \\ [\circ,\psi] & 0 & 0 & 0 \\ [\circ,gB_h] & [\circ,\psi] & 0 & 0 \end{pmatrix}, \qquad (6)$$

where the Poisson bracket is defined by $[a,b] := \boldsymbol{h}\cdot\nabla a\times\nabla b$ for arbitrary functions a and b The evolution equation is given by $\partial v^i/\partial t = \mathcal{J}^{ij}\delta H/\delta v^j$.

Casimir invariants $C[v]$, that satisfy $\mathcal{J}^{ij}\delta C/\delta v^j = 0$, were found to be

$$C[v] = \int \mathrm{d}V\,\Big(UF_1(\psi) + \left(u_hF_1'(\psi) + F_2'(\psi)\right)\left(B_h^\star + f\psi\right) - fF_2(\psi) + u_hF_3(\psi) + F_4(\psi)\Big), \qquad (7)$$

where $F_i(\psi)$ $(i = 1,2,3,4)$ are arbitrary functions of ψ, and the prime denotes a derivative with respect to ψ.

Equilibria of the system can be obtained by setting the first variation of the energy-Casimir functional $F[v] := H[v] + C[v]$ zero[2]. Three of the four equations can be solved algebraically as

$$\varphi = F_1, \qquad (8)$$

$$u_h = \frac{1}{1-(F_1')^2}\left(-\frac{1}{g}F_3 + F_1'F_2'\right), \qquad (9)$$

$$B_h = \frac{1}{1-(F_1')^2}\left(\frac{1}{g}F_1'F_3 - F_2'\right), \qquad (10)$$

where $F_1' \neq \pm 1$ is assumed. The remaining equation can be summarized as

$$\left(1-(F_1')^2\right)\mathcal{L}\psi - F_1'F_1''\left(g\left(\frac{\partial\psi}{\partial r}\right)^2 + \frac{1}{K^2r^2}\left(\frac{\partial\psi}{\partial\alpha}\right)^2\right) = -fF_2' + F_4' + \left(\frac{F_1'F_2'F_3}{1-(F_1')^2}\right)' - \frac{1}{2}\left(\frac{g\,(F_2')^2 + \frac{1}{g}F_3^2}{1-(F_1')^2}\right)'. \qquad (11)$$

Equation (11) is an elliptic equation for $\psi(r,\alpha)$, and can be solved under an appropriate boundary condition such as given ψ_ℓ at a radial position. The solution gives a helically symmetric equilibrium.

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References

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