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Riemann solvers for MHD: 20 years of the HLLD solver and beyond

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Shocks and discontinuities are ubiquitous in laboratory, space, and astrophysical plasmas, playing crucial roles in various phenomena. In magnetohydrodynamics (MHD), a system that describes the macroscopic dynamics of plasmas, such discontinuous structures, which are much more complicated than those in neutral hydrodynamics, can naturally form. Capturing these complex MHD shocks and discontinuities sharply and accurately without numerical oscillations has long been a central challenge in computational MHD.

One of the most promising so-called "shock capturing" approaches is the class of Godunov-type methods, also known as Riemann solvers. In Riemann solvers, the numerical fluxes for the conservation laws are evaluated at cell interfaces from an approximate solution of the Riemann problem, which is an initial value problem whose initial conditions consist of two constant states separated by a single discontinuity. Therefore, the quality of the numerical solution to the conservation laws has a direct and significant impact on the accuracy of the approximate solution of the Riemann problem.

Over the decades, several Riemann solvers have been proposed for MHD. The Roe solver [1], which uses the exact solution of the locally linearized Riemann problem at the cell interfaces, has been applied to MHD [2]. The Roe solver can sharply resolve shocks and discontinuities by taking into account all characteristic waves. However, especially in MHD, it suffers from the computational expense associated with highly complicated eigenvectors and may fail to preserve the positivity of density and pressure.

The Harten-Lax-van Leer (HLL) solver [3], another simple solver, approximates the internal state in the Riemann fan between the two outermost waves as a single constant state. HLL-type solvers are constructed based on nonlinear jump conditions and are generally robust. However, because the HLL solver neglects the intermediate waves within the Riemann fan, it cannot sharply resolve discontinuities such as contact and shear waves. To overcome these limitations, an epoch-making Riemann solver called the HLL-Discontinuities (HLLD) solver [4] was proposed. In the HLLD solver, the normal velocity inside the Riemann fan is assumed to be constant, and multiple intermediate constant states are then analytically determined by satisfying the nonlinear jump conditions for MHD. While preserving the positivity of density and pressure, the HLLD solver, unlike the original HLL solver, can exactly resolve isolated contact and rotational discontinuities. Thus, the HLLD solver achieves high accuracy, robustness, and computational efficiency.

In the 20 years since the HLLD solver was proposed, the HLLD solver has been adopted as the de facto standard solver in many open-source MHD codes [5] worldwide, significantly contributing to the understanding of various phenomena, particularly in space and astrophysical plasmas. Although the original HLLD solver was designed for fully compressible and non-relativistic MHD, its application has been successfully extended to isothermal MHD [6], relativistic MHD [7], and MHD with Boris correction [8]. Furthermore, the underlying concept of the HLLD solver has even been applied beyond MHD, including to problems in elastic-plastic solid mechanics [9].

Despite its widespread use and various extensions, the HLLD solver still faces challenges in extreme parameter regimes. At very low Mach numbers, it suffers from excessive numerical viscosity, as with other Riemann solvers. Conversely, in multi-dimensional flows at very high Mach numbers, numerical shock instabilities can also appear, as is the case with other high-resolution solvers. To partly address these issues, an "all-speed" variant of the HLLD solver was proposed [10]. Nevertheless, achieving accurate solutions for very low-beta MHD remains particularly difficult. This difficulty may be related to the issue of "energy consistency" discussed in [11]. Developing an "all-beta" variant of the HLLD solver that remains robust across the full range of plasma beta is an important open problem and one of the promising directions for future research.

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