Theory of Wave Turbulence

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The theory of wave turbulence [1]—developed over the past five decades—is a consistent framework for describing cascades in a broad class of weakly interacting systems including: waves in the ocean, plasma waves, spin waves, acoustic waves, and many others. The Kolmogorov-Zakharov scaling $n_k \sim k^{-\gamma}$ for the distribution of mode occupation numbers is an exact solution of the weakly interacting kinetic equations, which been verified numerically and/or experimentally in a number of contexts [2,3].

If the flux pumped into the system is large, or even if the flux is small and one is far along the cascade (at a wavenumber that differs significantly from the pumping scale)—as is often the case in physical realizations—the standard weak wave turbulence theory is insufficient. Recent progress in kinetic theory has provided a systematic framework for computing the kinetic equation, perturbatively in the interaction [4].

We go beyond Kolmogorov-Zakharov scaling in the context of an inverse cascade in the nonlinear Schrodinger equation, a universal model describing spectrally narrow wavepackets [5]. While the focusing and defocusing nonlinear Schrodinger equations have similar behavior in the weak turbulence regime, they must differ dramatically in the strong turbulence regime. We show that this difference is already present at next-to-leading order in the nonlinearity in the weak turbulence regime: The one-loop correction to the interaction vertex suppresses repulsion (like screening in QED), leading to a steeper spectrum in the defocusing case. In contrast, attraction enhancement (like antiscreening in QCD) makes the spectrum less steep in the focusing case.

To describe strong turbulence, we consider a vector model in the limit of a large number of components. A kinetic equation, valid at all scales, can be derived analytically. It has an inverse-cascade solution whose two asymptotics, at high and low wavenumbers, describe weak and strong turbulence, respectively. We find two forms of universality in the strong turbulence spectrum: in focusing media it is independent of the flux magnitude (the widely used critical balance solution), while in defocusing media it is independent of the bare coupling constant, with the largest scale appearing instead.

We will describe analogous results for turbulence of spin waves in ferromagnets. We expect similar behavior for an inverse cascade in an isothermal plasma; going beyond the weak turbulence approximation, even for weak nonlinearity, may be necessary for thermonuclear studies.

References

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