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Mixing in a two-dimensional fluid and the curvature of the flow domain: how to make the vorticity field evolve towards the statistical equilibrium

Koki Ryono[1], Keiichi Ishioka[1]
[1] Division of Earth and Planetary Sciences, Graduate School of Science, Kyoto University
e-mail (speaker): ryono@kugi.kyoto-u.ac.jp

The mixing of the vorticity field in two-dimensional turbulence is considered a key process in the formation of large, coherent structures. The degree of the mixing can be quantified by using the mixing entropy, which is defined in the framework of statistical mechanics (e.g., [1]). If mixing is ideal, the vorticity field evolves towards the statistical equilibrium which is defined as the maximizer of the entropy. However, it is known that the dynamics governed by the Euler equation does not necessarily reach the statistical equilibrium (e.g., [2]).

In systems in which the vorticity or the potential vorticity is materially conserved, such as the two-dimensional Euler equation or the Charney-Hasegawa-Mima equation, mixing occurs through vorticity filamentation. We recently showed that the negative curvature of the flow domain accelerates the elongation of a material line in the fluid [3]. This suggests that the Riemannian metric may influence the relaxation of the vorticity field towards the statistical equilibrium.

In this study, we test the relationship between the curvature of the domain and mixing of the vorticity field by numerical calculations. We consider a doubly 2π-periodic torus $M = (\mathbf{R}/2\pi\mathbf{Z})^2$ equipped with a time-periodic Riemannian metric

$$g(t) = \begin{pmatrix} \gamma(x, y; t) & 0 \\ 0 & \gamma(x, y; t)^{-1} \end{pmatrix}$$

where x and y are the standard coordinates of the torus, and γ is a positive function. Since $\det g(t) = 1$, the deformation preserves the total area of the torus.

The Gaussian curvature R of the torus is given by

$$R = -\frac{1}{2}\left\{\frac{\partial^2}{\partial x^2}\left(\frac{1}{\gamma}\right) + \frac{\partial^2 \gamma}{\partial y^2}\right\}$$

and the area element remains $dxdy$. Similar to Euler flows on flat domains, we consider the vorticity equation

$$\frac{\partial q}{\partial t} + \frac{\partial \psi}{\partial x}\frac{\partial q}{\partial y} - \frac{\partial q}{\partial x}\frac{\partial \psi}{\partial y} = 0$$

where q is the vorticity and ψ is the stream function determined by

$$q = \Delta_{g(t)}\psi := \frac{\partial}{\partial x}\left(\frac{1}{\gamma}\frac{\partial \psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\gamma \frac{\partial \psi}{\partial y}\right).$$

Note that the Laplace-Beltrami operator $\Delta_{g(t)}$ is time-dependent. The system conserves the Casimir invariants

$$C_f := \int_0^{2\pi}\int_0^{2\pi} f\big(q(x, y)\big)\,dx\,dy$$

for any function f, although the energy is not conserved.

For numerical calculations, we consider the case in which γ is independent of x. Furthermore, we introduce a new latitudinal coordinate $Y = Y(y, t)$ which fulfills a differential equation

$$\frac{\partial y}{\partial Y} = \gamma(x, y; t),$$

when $y = y(Y, t)$ is viewed as a function of the new coordinates Y and t. We impose a periodicity condition $y(Y + 2\pi, t) = y(Y, t) + 2\pi$. Then, the Euler equation in the coordinate system (x, Y, t) becomes

$$\frac{\partial \tilde{q}}{\partial t} - \frac{\dot{y}}{\tilde{\gamma}}\frac{\partial \tilde{q}}{\partial Y} + \frac{1}{\tilde{\gamma}}\left(\frac{\partial \tilde{\psi}}{\partial x}\frac{\partial \tilde{q}}{\partial Y} - \frac{\partial \tilde{q}}{\partial x}\frac{\partial \tilde{\psi}}{\partial Y}\right) = 0$$

and the Laplace-Beltrami operator can be written as

$$\Delta_{g(t)}\tilde{\psi} = \frac{1}{\gamma}\left(\frac{\partial^2 \tilde{\psi}}{\partial x^2} + \frac{\partial^2 \tilde{\psi}}{\partial Y^2}\right),$$

where tilde denotes a function expressed in the new coordinates (x, Y, t), and $\dot{y} = (\partial y/\partial t)(Y, t)$. The Euler equation in this form can be numerically integrated by using the spectral method.

In the talk, we will show some results from time integrations and analyze how domain geometry affects vorticity filamentation, mixing, and entropy production. Since the energy is not conserved, the flows cannot be directly compared to those on a flat torus or statistical equilibria. However, by incorporating artificial dynamics such as [4], the energy can be restored. Furthermore, the dynamics on a periodically deforming torus may be an interesting object for the non-equilibrium statistical mechanics. On a flat rectangular domain, the energy of the Euler flow can be extracted by periodically deforming the boundary of the domain, which is related to the negativeness of the temperature in two-dimensional turbulence [5]. As future work, we aim to investigate the relationship between the curvature of the domain and the amount of energy extracted.

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