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Nambu Bracket, isomagnetovortical perturbations and wave energy for compressible baroclinic MHD

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1. Introduction: Incompressible barotropic flows

The dynamics of an inviscid incompressible fluid is described by the Hamiltonian equation for a functional of the velocity $\mathbf{v}$ with respect to the Lie-Poisson bracket:

$$\frac{dF}{dt} = \{F, H\};\ \{F, H\} := \langle\left[\frac{\delta F}{\delta v}, \frac{\delta H}{\delta v}\right], \boldsymbol{v}\rangle$$

The Nambu-bracket [1] manifests the helicity h, a topological invariant, lying behind it.

$$\{F, H\} = \int \frac{\delta h}{\delta \boldsymbol{v}} \cdot \left(\frac{\delta F}{\delta \boldsymbol{v}} \times \frac{\delta H}{\delta \boldsymbol{v}}\right) d^3x := \{F, h, H\}; \qquad (1)$$

$$h = \frac{1}{2}\int \boldsymbol{v} \cdot (\nabla \times \boldsymbol{v}) d^3x.$$

A direct consequence of (1) is the Euler equation

$$\frac{\partial \boldsymbol{v}}{\partial t} = \wp\left[\frac{\delta H}{\delta \boldsymbol{v}} \times \frac{\delta h}{\delta \boldsymbol{v}}\right], \qquad (2)$$

where $\wp[\cdot]$ is the operator projecting to solenoidal vector field. An *isovortical perturbation* $\delta \boldsymbol{v}$ is a vector field that lies on $h = \text{const.}$, and is provided by leaving the Hamiltonian an arbitrary functional K,

$$\delta \mathbf{v} = \wp\left[\boldsymbol{\xi} \times \frac{\delta h}{\delta \boldsymbol{v}}\right], \qquad (3)$$

where $\boldsymbol{\xi} = \frac{\partial K}{\delta v}$ is an arbitrary vector field. Arnold theorem [2] states that a steady state $\boldsymbol{v}$, of vanishing RHS of (2),.is characterized as an extremal of the kinetic energy H with respect to the isovortical ones (3).

$$\delta H = \int \frac{\delta H}{\delta \boldsymbol{v}} \cdot \delta \boldsymbol{v} d^3x = \int \boldsymbol{\xi} \cdot \left(\frac{\delta h}{\delta \boldsymbol{v}} \times \frac{\delta H}{\delta \boldsymbol{v}}\right) d^3x = 0. \quad (4)$$

Knowledge of stability and bifurcation of a flow is gained from the spectra of the linearized Euler equation. According to Krein's theory of Hamiltonian spectra, the signature of wave energy plays a vital role for the stability criterion; coexistence of two modes with opposite signed energy or of zero-energy modes is necessary for triggering instability [2,3]. Owing to the critical property (4), the energy, of second order in amplitude, of an isovortical perturbation is expressible solely in terms of first-order perturbation $\boldsymbol{\xi}$.

$$\delta^2 H = \frac{1}{2}\int \boldsymbol{\omega} \cdot \left(\frac{\partial \boldsymbol{\xi}}{\partial t} \times \boldsymbol{\xi}\right) d^3x, \qquad (5)$$

where $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$ is the vorticity field [2,4].

In this investigation, we extend the energy formula (5) to compressible non-isentropic magnetohydrodynamics (MHD) [5].

2. Nambu bracket for compressible baroclinic MHD

Motion of the compressible non-isentropic MHD is governed by the equation of continuity, the Euler equations augmented by the Lorentz force, the adiabatic condition and the induction equation for the density $\rho(\mathbf{x}, t)$, the velocity field $\boldsymbol{v}(\mathbf{x}, t)$, the specific entropy $s(\mathbf{x}, t)$ and the magnetic field $\mathbf{B}(\mathbf{x}, t)$.

The total mass $M = \int \rho d^3x$, the total entropy $S = \int \rho s d^3x$, the magnetic helicity $h_m = \int \mathbf{A} \cdot \mathbf{B} \boldsymbol{d}^3 x$ with $\mathbf{B} = \nabla \times \mathbf{A}$, and the cross helicity $h_c = \int \mathbf{M} \cdot \mathbf{d} d^3x$, with $\mathbf{M} = \rho \mathbf{v}$ and $\mathbf{d}$ being the displacement field, constitute a complete set of the Casimirs. The Nambu bracket (1) is then extended to

$$\frac{dF}{dt} = \{F, h_c, H\}_{MMd} + \{F, S, H\}_{M\rho s} + \{F, h_m, H\}_{MBB}$$

the first bracket of which takes, for instance,

$$\{F, h_c, H\}_{MMd} = -\int \{\frac{\delta h_c}{\delta \boldsymbol{d}} \cdot [\frac{\delta F}{\delta \boldsymbol{M}} \cdot \nabla \ \frac{\delta H}{\delta \boldsymbol{M}} - \ \frac{\delta H}{\delta \boldsymbol{M}} \cdot \nabla \ \frac{\delta F}{\delta \boldsymbol{M}}] + \text{cyc}(F, h_c, H)\} d^3x$$

with $\text{cyc}(\cdot,\cdot,\cdot)$ signifying the cyclic permutations [1].

3. Wave energy for compressible baroclinic MHD

Perturbations preserving all the Casimir invariants are referred to as the *isomagnetovortical perturbations* in the context of MHD [5], and are generated by leaving the Hamiltonian an arbitrary functional $K(\mathbf{M}, \rho, s, \mathbf{B})$. By restricting to this class of waves, calculation of their energy becomes feasible, without having to necessitate perturbations of higher than the first order in amplitude.

To perform calculation of the wave energy, it is judicious to start from the Frieman-Rotenberg equation, governing the Lagrangian displacement field $\boldsymbol{\xi}(\mathbf{x}, t)$.

$$\rho \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} + 2\rho(v \cdot \nabla)\frac{\partial \boldsymbol{\xi}}{\partial t} = \boldsymbol{F}(\boldsymbol{\xi}),$$

where $\mathbf{F}(\boldsymbol{\xi})$ is the force operator of thermodynamic as well as hydrodynamic origin. For our purpose, we have to confirm that $\mathbf{F}$ is self-adjoint, which is a bit of task. Self-adjointness of $\mathbf{F}$ guarantees that the energy of waves is calculated through

$$\delta^2 H = \frac{1}{2}\int \rho \left\{\left(\frac{\partial \boldsymbol{\xi}}{\partial t}\right)^2 - \boldsymbol{\xi} \cdot \boldsymbol{F}(\boldsymbol{\xi})\right\} d^3x. \qquad (6)$$

Starting from (6), we manipulate the energy formula extending (5) to allow for the compressibility and baroclinic effect. The resulting formula is compared with other formulas.

References

[1] Y. Fukumoto and R. Zou: Prog. Theor. Exp. Phys. (PTEP). **7**, 8 (2024).
[2] V. I. Arnold: Ann. Inst. Fourier Grenoble **16,** 319-361 (1966).
[3] Y. Fukumoto: J. Fluid Mech. **493**, 287-318 (2003).
[4] M. Hirota and Y. Fukumoto, J. Math. Phys. **49**, 083101 (2008)
[5] Y. Fukumoto and R. Zou: Rev. Mod. Plasma Phys. **7**, 8 (2003).